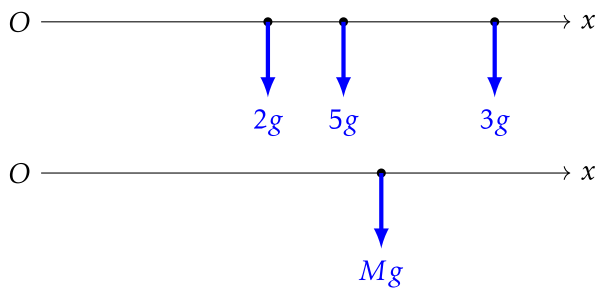
Systems of Particles

Example

Suppose we have three particles with masses 2 kg, 5 kg, and 3 kg which lie on the x-axis at the points $(3,0)$, $(4,0)$ and $(6,0)$. Find a single force which is equivalent to the weight of these three particles.



$$\begin{aligned} N2(\downarrow): \quad & Mg = 2g + 5g + 3g \\ \Rightarrow \quad & M = 10 \\ \overset{\frown}{O}: \quad & Mg \cdot \overline{x} = 2g \cdot 3 + 5g \cdot 4 + 3g \cdot 6 \\ \Rightarrow \quad & \overline{x} = \frac{2 \cdot 3 + 5 \cdot 4 + 3 \cdot 6}{10} \\ & = 4.4 \end{aligned}$$

Fact (Centre of mass of a system of particles) — For a system of particles with positions $\mathbf{x}_i$ and masses m_i, they are equivalent to a particle with mass $\sum m_i$ and position

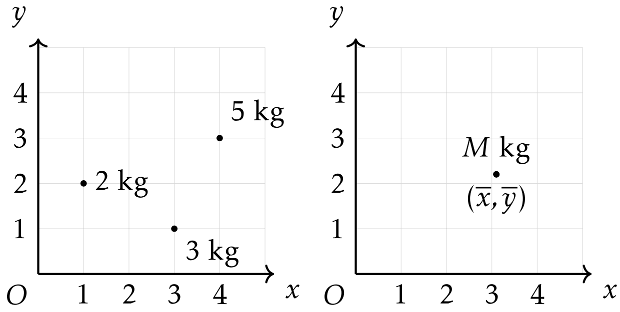
$$\left(\sum_i m_i\right)\overline{\mathbf{x}} = \sum_i m_i \mathbf{x_i}$$

Tip

We can do everything in vector form, but sometimes it's easier to do everything component wise.

Example

Find the coordinates of the centre of mass of the following system of particles: 2 kg at $(1,2)$; 3 kg at $(3,1)$; 5 kg at $(4,3)$;



$$
\begin{aligned}
\left(\sum_i m_i\right)\overline{\mathbf{x}} &= \sum_i m_i \mathbf{x}_i \\
10\overline{\mathbf{x}} &= 2\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}3\\1\end{pmatrix} + 5\begin{pmatrix}4\\3\end{pmatrix} \\
&= \begin{pmatrix}2+9+20\\4+3+15\end{pmatrix} \\
&= \begin{pmatrix}31\\22\end{pmatrix} \\
\Rightarrow \quad \overline{x} &= \begin{pmatrix}3.1\\2.2\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\left(\sum_i m_i\right)\overline{x} &= \sum_i m_i x_i \\
10\overline{x} &= 2\cdot 1 + 3\cdot 3 + 5\cdot 4 \\
&= 31 \\
\Rightarrow \quad \overline{x} &= 3.1 \\
\left(\sum_i m_i\right)\overline{y} &= \sum_i m_i y_i \\
10\overline{y} &= 2\cdot 2 + 3\cdot 1 + 5\cdot 3 \\
&= 22 \\
\Rightarrow \quad \overline{y} &= 2.2
\end{aligned}
$$

Example

Three particles of mass 2 kg, 1 kg and $m$kg are situated at the points $(-1,3)$, $(2,9)$ and $(2,-1)$ respectively. Given that the centre of mass of the three particles is at $(1,\overline{y})$, find

(a) the value of m,

(b) the value of $\overline{y}$.

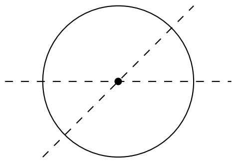
$$
\begin{aligned}
2\begin{pmatrix}-1\\3\end{pmatrix} + 1\begin{pmatrix}2\\9\end{pmatrix} + m\begin{pmatrix}2\\-1\end{pmatrix} &= (2+1+m)\begin{pmatrix}1\\\overline{y}\end{pmatrix} \\
\Rightarrow \quad \begin{pmatrix}2m\\15-m\end{pmatrix} &= \begin{pmatrix}3+m\\(3+m)\overline{y}\end{pmatrix} \\
\Rightarrow \quad m &= 3, \overline{y} = 2
\end{aligned}
$$

Standard Shapes

Fact — If a shape has a line of symmetry, the centre of mass lies on that line.

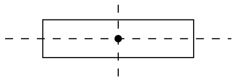
We are also interested

Circle



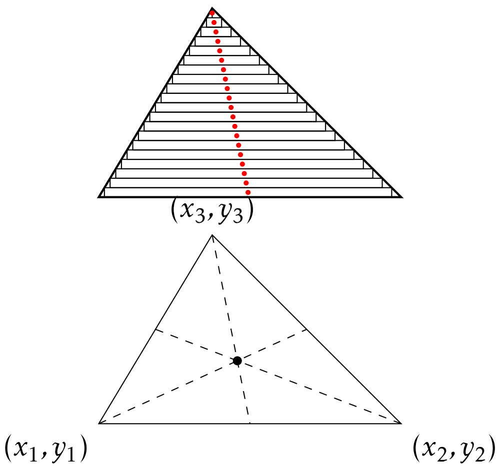
Circles clearly have many lines of symmetry, all meeting at the centre, so the centre of mass of a circle is its centre!

Rectangle



Rectangles have (at least) two lines of symmetry meeting at the centre, so the centre of mass is also at its centre.

Triangle



Imagine drawing a series of rectangles inside your triangle, and the triangle being a combination of these rectangles. The C.O.M will have to lie on the line of these rectangles. *[If this argument makes you nervous it should!]*

These lines (they are called *medians*) all meet at a point, this is the centre of mass. We can calculate this point easily:

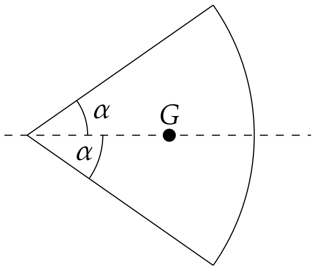
$$(\overline{x}, \overline{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

Example

A uniform triangular lamina has vertices $A(1,4), B(3,2)$ and $C(5,3)$. Find the coordinates of its centre of mass.

G is the point $\left(\frac{1+3+5}{3}, \frac{4+2+3}{3}\right) = (3,3)$

Sector of a circular disc



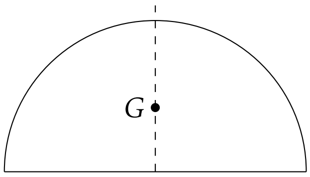
For a uniform sector of a circle, of radius r and centre angle 2α, the centre of mass will be on the axis of symmetry, a distance of $\frac{2r\sin\alpha}{3\alpha}$ from the centre.

Tip

Do not forget the factor of two in the angle!

Example

How far is the centre of mass from the centre of a uniform semi-circular lamina?

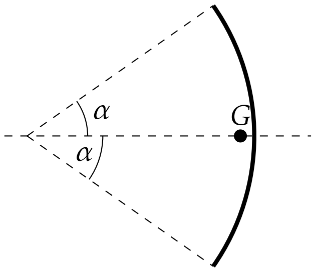


The centre of mass must lie on the dotted line. The distance from the centre must be:

$$\frac{2\cdot r\cdot \sin\alpha}{3\alpha} = \frac{2r\sin\frac{\pi}{2}}{3\frac{\pi}{2}}$$
$$= \frac{4r}{3\pi}$$

Ie the centre is $\frac{4}{3\pi}$ of a radius from the centre.

Arc of a uniform wire

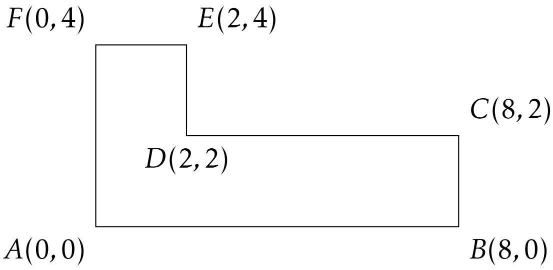


For a uniform wire bend into a circular arc, with central angle 2α, the centre of the mass is $\frac{r\sin\alpha}{\alpha}$ from the centre of the circle.

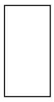
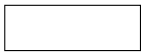
Composite Shapes

Example

The diagram shows a uniform lamina. Find the coordinates of the centre of mass.



Method 1

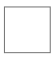
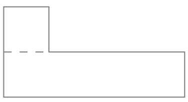
			
Area	8	12	20
x	1	5	$\overline{x}$
y	2	1	$\overline{y}$

$$8\begin{pmatrix}1\\2\end{pmatrix}+12\begin{pmatrix}5\\1\end{pmatrix}=20\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix} \qquad (\textstyle\sum m_i\mathbf{x}_i=(\sum m_i)\overline{\mathbf{x}})$$

$$\Rightarrow \begin{pmatrix}68\\28\end{pmatrix}=20\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}=\begin{pmatrix}3.4\\1.4\end{pmatrix}$$

Method 2

			
Area	4	16	20
x	1	4	$\overline{x}$
y	3	1	$\overline{y}$

$$4\begin{pmatrix}1\\3\end{pmatrix}+16\begin{pmatrix}4\\1\end{pmatrix}=20\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix} \qquad (\textstyle\sum m_i\mathbf{x}_i=(\sum m_i)\overline{\mathbf{x}})$$

$$\Rightarrow \begin{pmatrix}68\\28\end{pmatrix}=20\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}=\begin{pmatrix}3.4\\1.4\end{pmatrix}$$

Method 3

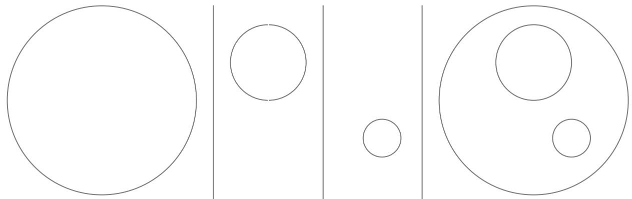
Area	32	12	20
x	4	5	$\overline{x}$
y	2	3	$\overline{y}$

$$32\begin{pmatrix}4\\2\end{pmatrix} - 12\begin{pmatrix}5\\3\end{pmatrix} = 20\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}68\\28\end{pmatrix} = 20\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix} = \begin{pmatrix}3.4\\1.4\end{pmatrix}$$

> **Example**
> A uniform circular disc, centre O, of radius 5 cm has two circular holes cut in it. A larger hole has radius 2 cm and centre $(0,2)$, a smaler hole of radius 1 cm is cut at $(2,-2)$. Find the coordinates of the centre of mass



Area	$\pi\cdot 5^2$	$\pi\cdot 5^2$	$\pi\cdot 5^2$	$\pi(5^2-2^2-1^2)$
x	0	0	2	$\overline{x}$
y	0	2	-2	$\overline{y}$

$$\cancel{\pi}\cdot 5^2\begin{pmatrix}0\\0\end{pmatrix} - \cancel{\pi}\cdot 2^2\begin{pmatrix}0\\2\end{pmatrix} - \cancel{\pi}\cdot 1^2\begin{pmatrix}2\\-2\end{pmatrix} = \cancel{\pi}(5^2-2^2-1^2)\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}-2\\-6\end{pmatrix} = 20\begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}\overline{x}\\\overline{y}\end{pmatrix} = \begin{pmatrix}-0.1\\-0.3\end{pmatrix}$$

Equilibrium of a rigid body

Hanging from a pivot

- In the first diagram, while forces are balanced, there is a turning movement about the pivot.
- In the second diagram, forces are balanced **and**, there is no turning moment.

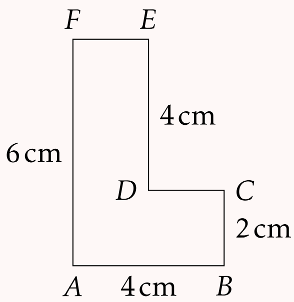
Example

Find the angle that the line AB makes with the vertical if this L-shaped uniform lamina is freely suspended from:

(a) A,

(b) B,

(c) E,

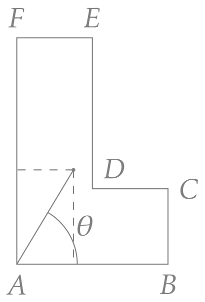


First, find the centre of mass:

$$8\begin{pmatrix}2\\1\end{pmatrix} + 8\begin{pmatrix}1\\4\end{pmatrix} = 16\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix} = \begin{pmatrix}1.5\\2.5\end{pmatrix}$$

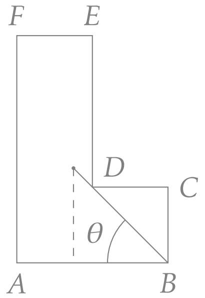
(a)



$$\tan\theta = \frac{2.5}{1.5}$$

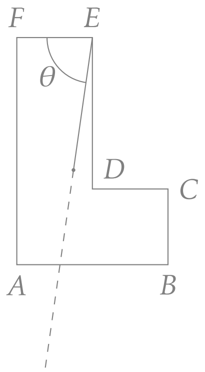
$$\Rightarrow \quad \theta = 59.0^\circ\ (3\,s.f.)$$

(b)



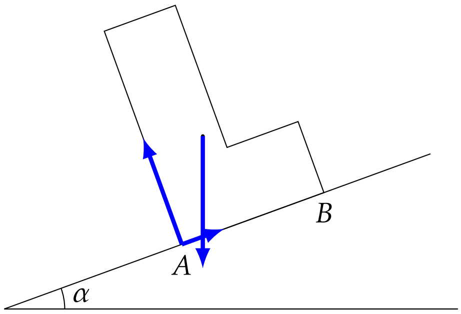
$$\tan\theta = \frac{2.5}{4-1.5} = 1$$
$$\Rightarrow \quad \theta = 45^\circ$$

(c)

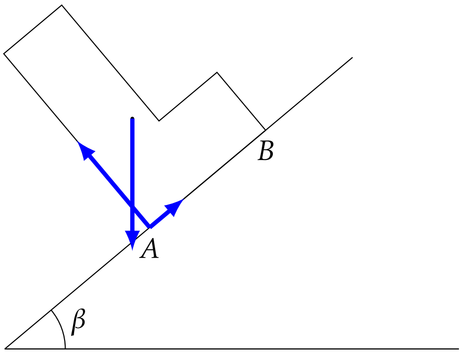


$$\tan\theta = \frac{6-2.5}{2-1.5} = 7$$
$$\Rightarrow \quad \theta = 81.9^\circ$$

Toppling Objects



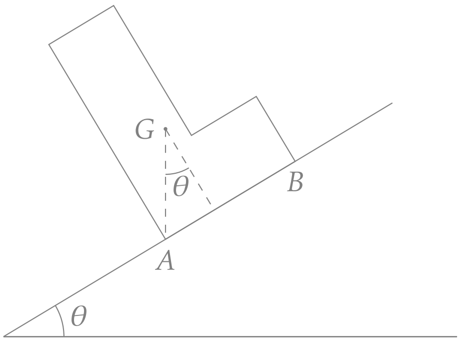
If we have an object resting on a plane, with its centre of mass directly over the base. There will be a turning moment into the slope.



On the other hand, if the line of action is outside the base, there will be a turning moment around the furthest point, causing the object to topple.

Example

Assuming the slope is rough enough to prevent sliding, what is the critical angle where the L-shaped lamina from before is just about to topple over?



When the lamina is about to topple, its centre of mass G will be vertically above the point A.

$$\tan\alpha = \frac{1.5}{2.5} = \frac{3}{5}$$

$$\Rightarrow \alpha = 31^\circ \text{ (nearest degree)}$$

Toppling vs Sliding

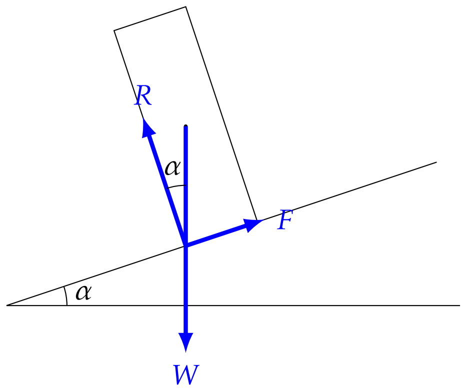
Example

A uniform solid cylinder is resting in equilibrium with its end on a rough plane inclined at a variable angle α to the horizontal. The cylinder has diameter and height 0.6 m and height 1.8 m.

(a) Assuming the plane is sufficiently rough to prevent sliding, find the maximum value of α which would allow the cylinder to continue to rest in equilibrium

The coefficient of friction between the cylinder and the plane is $\frac{2}{9}$.

(b) Find the angle at which the cylinder starts to slide. Show that the cylinder slides before it topples.



(a) The limiting point will be when the centre of mass is directly over the corner. The centre of mass is 0.3 m *along, and*0.9 m *above the base. Therefore*

$$\tan\alpha = \frac{0.3}{0.9}$$
$$\Rightarrow \quad \alpha = 18.4^\circ$$

(b)

$$N2(\nwarrow): \quad R - W\cos\alpha = 0$$
$$\Rightarrow \quad R = W\cos\alpha$$
$$N2(\nearrow): \quad W\sin\alpha - F = ma$$
$$(F = \mu R): \quad W\sin\alpha - \mu R > 0$$
$$\Rightarrow \quad W\sin\alpha - \frac{2}{9}W\cos\alpha > 0$$
$$\Rightarrow \quad \tan\alpha > \frac{2}{9}$$
$$\Rightarrow \quad \alpha > 12.5^\circ$$

Therefore we start sliding when $\alpha > 12.5^\circ$, but only start toppling when $\alpha > 18.4^\circ$. Therefore we slide before toppling.